Problem 1.8

For a solid, we also define the **linear thermal expansion coefficient**, α, as the fractional increase in length per degree:

$$\alpha \equiv \frac{\Delta L/L}{\Delta T}.$$

(a) For steel, α is $1.1 \times 10^{-5}\ \mathrm{K}^{-1}$. Estimate the total variation in length of a 1-km steel bridge between a cold winter night and a hot summer day.

(b) The dial thermometer in Figure 1.2 uses a coiled metal strip made of two different metals laminated together. Explain how this works.

(c) Prove that the volume thermal expansion coefficient of a solid is equal to the sum of its linear expansion coefficients in the three directions: $\beta = \alpha_x + \alpha_y + \alpha_z$. (So for an isotropic solid, which expands the same in all directions, $\beta = 3\alpha$.)

Solution

Part (a)

The total variation in length is what ΔL represents. Solve the given equation for it.

$$\Delta L = \alpha L \Delta T$$

Assume that the hottest summer day is 70°C and that the coldest winter day is −30°C so that the temperature difference is $\Delta T = 100$ K. The total variation in length is then

$$\begin{aligned}\Delta L &= (1.1 \times 10^{-5}\ \mathrm{K}^{-1})(1\ \mathrm{km})(100\ \mathrm{K}) \\ &= 1.1 \times 10^{-3}\ \mathrm{km} \\ &= 1.1\ \mathrm{m}.\end{aligned}$$

Therefore, the bridge will elongate about one meter at most from the coldest to the hottest times.

Part (b)

The two different metals have different linear thermal expansion coefficients. Consequently, one metal elongates (shortens) more than the other does for a given temperature increase (decrease). This results in the turning of the metal coil, which the thermometer measures.

Part (c)

Suppose there's a rectangular solid with length x, width y, and height z at some temperature T. Suppose also that at some higher temperature $T + \Delta T$, the solid has length $x + \Delta x$, width $y + \Delta y$, and height $z + \Delta z$ as a result of thermal expansion.

The change in volume between these two temperatures is

$$\begin{aligned}\Delta V &= (x+\Delta x)(y+\Delta y)(z+\Delta z) - xyz \\ &= (xy + x\Delta y + y\Delta x + \Delta x\Delta y)(z+\Delta z) - xyz \\ &= \cancel{xyz} + xz\Delta y + yz\Delta x + z\Delta x\Delta y + xy\Delta z + x\Delta y\Delta z + y\Delta x\Delta z + \Delta x\Delta y\Delta z - \cancel{xyz} \\ &= xz\Delta y + yz\Delta x + \underbrace{z\Delta x\Delta y}_{\text{negligible}} + xy\Delta z + \underbrace{x\Delta y\Delta z}_{\text{negligible}} + \underbrace{y\Delta x\Delta z}_{\text{negligible}} + \underbrace{\Delta x\Delta y\Delta z}_{\text{very negligible}}.\end{aligned}$$

The changes in length due to thermal expansion are generally very small, that is, $\Delta x \ll 1$ and $\Delta y \ll 1$ and $\Delta z \ll 1$. That means terms with more than one difference are negligible compared to those with only one. To a good approximation, then,

$$\begin{aligned}\Delta V &= xz\Delta y + yz\Delta x + xy\Delta z \\ &= xyz\left(\frac{\Delta y}{y} + \frac{\Delta x}{x} + \frac{\Delta z}{z}\right) \\ &= V\left(\frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z}\right). \qquad (1)\end{aligned}$$

The given definition for the linear thermal expansion coefficient applies in each dimension.

$$\alpha = \frac{\Delta L/L}{\Delta T} = \frac{\Delta L}{L\Delta T} \quad \Rightarrow \quad \begin{cases}\alpha_x = \dfrac{\Delta x}{x\Delta T} \\ \alpha_y = \dfrac{\Delta y}{y\Delta T} \\ \alpha_z = \dfrac{\Delta z}{z\Delta T}\end{cases}$$

As a result, equation (1) becomes

$$\begin{aligned}\Delta V &= V(\alpha_x\Delta T + \alpha_y\Delta T + \alpha_z\Delta T) \\ &= (\alpha_x + \alpha_y + \alpha_z)V\Delta T.\end{aligned}$$

Comparing this to the definition of β in Problem 1.7,

$$\Delta V = \beta V\Delta T,$$

we have

$$\beta = \alpha_x + \alpha_y + \alpha_z.$$

Therefore, the volume thermal expansion coefficient of a solid is equal to the sum of its linear expansion coefficients in the three directions. An isotropic solid expands the same in all directions ($\alpha_x = \alpha_y = \alpha_z = \alpha$), giving

$$\beta = 3\alpha.$$